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Line bundles over families of (super) Riemann surfaces. II: The graded case *

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A relative Picard theory in the context of graded manifolds is introduced. A Berezinian calculus and a theory of connections over SUSY-curves are systematically developed, and used to prove a Gauss–Bonnet theorem for line bundles in that setting and to discuss the validity of a flatness theorem.

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1. Introduction

From a geometric point of view, the application of moduli space techniques to the computation of quantum scattering amplitudes à la Polyakov for superstrings requires the introduction of the notion of super Riemann surface (also called SUSY-curve) [13]. These structures are most conveniently studied within the framework of Berezin–Leĭtes–Kostant's graded manifolds; physics suggests to define a SUSY-curve as a family of complex graded manifolds of relative dimension (1,1), additionally endowed with a *conformal structure.*

In this paper we complete the work we started in ref. [5], where we considered some facts concerning line bundles over families of ordinary Riemann surfaces. Here we prove a version of the Gauss–Bonnet theorem suitable to the context

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of SUSY-curves, and discuss the validity of a flatness theorem for SUSY-curves (cf. refs. [2,6]). With "flatness theorem" we refer to the classical result that a holomorphic line bundle on a Riemann surface is flat if and only if its Chern class vanishes. One should notice that the conformal structure of a SUSY-curve, as opposed to families of (1,1) dimensional complex graded manifolds without additional structures, provides essential tools for dealing with these problems.

The contents of this paper are as follows. In section 2 we present the extension to the graded case of some concepts pertaining to families of manifolds; this includes a relative graded de Rham theory, the notions of a relative Berezinian sheaf and fiberwise Berezin integration, a relative graded Serre duality, and a suitable relative Picard theory. In section 3, after recalling the definition of SUSY-curve, we show how on such objects a Berezinian differential calculus can be developed; one can even introduce a relative de Rham–Berezin theory.

The last section is devoted to the demonstration of the announced results; in particular, the statement of the Gauss–Bonnet theorem requires the consideration of *conformal connections* over the SUSY-curve. The transposition of the flatness theorem to the graded setting requires some care. Indeed, under very general assumptions, which are, for instance, satisfied by the moduli space of SUSY-curves, the relative flatness of a relative line bundle is sufficient, but not necessary, to ensure that it has vanishing relative Chern class. In order that relative flatness is also a necessary condition one needs stronger assumptions, as we shall discuss in detail.

Some of these notions and results already appeared in ref. [6], even though the treatment given there is less systematic and precise; for instance, the transition from the absolute to the relative case is not obtained simply by replacing the sheaf cohomology functors by the higher direct image functors. The case of a family of graded manifolds is intrinsically different from the case of a single graded manifold, basically because there is no notion of "local splitness" which is compatible with the family structure [16,20]. Thus, several concepts in complex geometry do not seem to admit a straightforward generalization to this setting, as for instance Grauert's cohomology base change theorem.

2. Graded families and line bundles

2.1. GRADED COMPLEX FAMILIES

Let us start by stating some facts about graded manifolds. We shall denote by $(X, \mathcal{B}_X)$ a complex analytic graded manifold [14], and by $(X, \mathcal{O}_X)$ the underlying complex analytic manifold. The structural epimorphism $\mathcal{B}_X \to \mathcal{O}_X$ induces an exact functor from the category of $\mathcal{B}_X$-modules to the category of $\mathcal{O}_X$-modules, which is realized by the tensor product $\otimes_{\mathcal{B}_X}\mathcal{O}_X$, and will be denoted by a tilde, i.e., $\widetilde{\mathcal{M}} \equiv \mathcal{M} \otimes_{\mathcal{B}_X} \mathcal{O}_X$.

A graded $\mathcal{B}_X$-module $\mathcal{M}$ is said to be *coherent* [17,19] if every $x \in X$ has a neighborhood U such that $\mathcal{M}|_U$ is finitely generated, and the kernel of any morphism $\mathcal{B}_X^{p|q}|_U \to \mathcal{M}|_U$ is locally finitely generated. The sheaves $\mathcal{B}_X^{p|q}$ are themselves coherent. If $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is a proper morphism of complex graded manifolds (which means that the underlying morphism $X \to Y$ is proper), and $\mathcal{M}$ is a coherent sheaf of graded $\mathcal{B}_X$-modules, then for all $k \geq 0$ the sheaf $R^k\pi_*\mathcal{M}$ (the kth higher direct image of $\mathcal{M}$) is a coherent graded $\mathcal{B}_Y$-module.

We say that a morphism $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is *flat* if $\pi_*\mathcal{B}_X$ is a flat graded $\mathcal{B}_Y$-module (then $\pi_*\mathcal{B}_X$ is locally free over $\mathcal{B}_Y$).

Definition 2.1. A family of complex analytic graded manifolds is a proper and flat submersion $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ such that the underlying morphism $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a family of complex manifolds. [#1]

This definition is tailored so as to include the moduli space of SUSY-curves (cf. ref. [10]). The structure sheaf $\mathcal{B}_{X_y}$ of the fiber over y is given by $\mathcal{B}_{X_y} = \mathcal{B}_X|_{X_y} \otimes_{(\mathcal{B}_Y)_y} k(y)$, with $k(y) = (\mathcal{B}_Y)_y/\mathfrak{m}_y \simeq \mathbb{C}$, where $\mathfrak{m}_y$ is the maximal ideal of $(\mathcal{B}_Y)_y$.

The *relative dimension* of the family is the pair of integers $(n, n') = \dim(X, \mathcal{B}_X) - \dim(Y, \mathcal{B}_Y)$, which is also the dimension of each fiber.

The following *vanishing theorem* holds.

Proposition 2.2. *Let $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ be a family of complex graded manifolds, of relative dimension (n, n'). Then $R^k\pi_*\mathcal{B}_X = 0$ for all $k > n$.*

Proof. Let $\mathcal{N}$ be the nilpotent ideal of $\mathcal{B}_Y$, and, for the sake of brevity, let us denote $\mathcal{M}_k = R^k\pi_*\mathcal{B}_X$. One can prove that $\widetilde{\mathcal{M}_k} \simeq R^k\pi_*\mathcal{O}_X$ by resolving $\mathcal{B}_X$ with an injective complex $\mathcal{I}^\bullet$ of $\mathcal{B}_X$-modules and noting that $\widetilde{\mathcal{I}}^\bullet$ is a resolution of $\mathcal{O}_X$ by injective abelian groups, which can be used to compute the sheaves $R^k\pi_*\mathcal{O}_X$ (cf. ref. [7], theorem 2.4.1). Now, $R^k\pi_*\mathcal{O}_X = 0$ for $k > n$ for the vanishing theorem for the underlying ordinary complex family, so that $\mathcal{M}_k/\mathcal{N}\mathcal{M}_k = 0$; since $\mathcal{B}_Y$ is a sheaf of local rings, and $\mathcal{M}_k$ is finitely generated as it is coherent, we may apply the graded Nakayama lemma [3] to obtain $\mathcal{M}_k = 0$ for $k > n$. □

One defines in the obvious way the sheaves of *relative graded derivations* and *relative graded differentials*, denoted by $\mathcal{D}er\,(\mathcal{B}_X/\mathcal{B}_Y)$ and $\Omega^1_{\mathcal{B}_X/\mathcal{B}_Y}$; for any fiber $(X_y, \mathcal{B}_{X_y})$, the sheaves $\Omega^1_{\mathcal{B}_{X_y}}$ of graded differentials are obtained from the

[#1] We recall from ref. [5] that a family of complex manifolds is a proper and flat submersion $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ whose fibers are universally connected.

sheaf of relative graded differentials as in the non-graded case, by restricting and tensoring by $k(y)$.

A similar definition of *family of real graded manifolds* can be given, and, as a matter of fact, any family of complex graded manifolds has an underlying real family $\pi : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ (for convenience, the sheaves $\mathcal{A}_X$ and $\mathcal{A}_Y$ are considered to be the complexifications of the structure sheaves of the relevant real graded manifolds). We denote by $\Omega^k_{\mathcal{A}_X/\mathcal{A}_Y}$ the kth exterior power of the sheaf $\Omega^1_{\mathcal{A}_X/\mathcal{A}_Y}$ of relative smooth graded differentials, and call *relative k-forms* its sections. One can introduce a *relative graded de Rham theory* as in the non-graded case [5], by considering a *relative graded differential* $d_r : \Omega^{k-1}_{\mathcal{A}_X/\mathcal{A}_Y} \to \Omega^k_{\mathcal{A}_X/\mathcal{A}_Y}$, and *relative graded de Rham sheaves* $DR^k_{\mathcal{A}_X/\mathcal{A}_Y} \equiv \pi_* \mathcal{Z}^k_{\mathcal{A}_X/\mathcal{A}_Y} / d_r \pi_* \Omega^{k-1}_{\mathcal{A}_X/\mathcal{A}_Y}$, where $\mathcal{Z}^k_{\mathcal{A}_X/\mathcal{A}_Y}$ denotes the closed relative k-forms (cf. ref. [18]). One can prove for these sheaves results analogous to the non-graded case: there is a sheaf isomorphism $DR^k_{\mathcal{A}_X/\mathcal{A}_Y} \simeq R^k \pi_* \pi^{-1} \mathcal{A}_Y$ (cf. ref. [16]), and the projection $p : \Omega^k_{\mathcal{A}_X} \to \Omega^k_{\mathcal{A}_X/\mathcal{A}_Y}$ induces a commutative diagram of $\mathbb{C}$-modules

$$\begin{array}{ccccc} \Gamma(X, \mathcal{Z}^k_{\mathcal{A}_X}) & \longrightarrow & H^k(X, \mathbb{C}) & \longrightarrow & \Gamma(Y, R^k \pi_* \mathbb{C}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ \Gamma(Y, \pi_* \mathcal{Z}^k_{\mathcal{A}_X/\mathcal{A}_Y}) & \longrightarrow & \Gamma(Y, DR^k_{\mathcal{A}_X/\mathcal{A}_Y}) & \xrightarrow{\sim} & \Gamma(Y, R^k \pi_* \pi^{-1} \mathcal{A}_Y) \end{array} \quad . \tag{2.1}$$

Remark 2.3. The relative graded de Rham sheaves do not coincide with the relative de Rham sheaves of the underlying ordinary family, contrary to the case of a single graded manifold, where the graded and ordinary de Rham theories do coincide [9]. This is in some sense reminiscent of what happens in the case of supermanifolds modelled over Grassmann algebras, cf. refs. [1,3].

A family of real graded manifolds is said to be *orientable* if the underlying family of differentiable manifolds is orientable; in this sense, any family of real graded manifolds underlying a family of complex analytic graded manifolds is canonically oriented.

2.2. FIBERWISE BEREZIN INTEGRATION

Let $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ be a family of complex analytic graded manifolds of relative dimension (n, n'). Let $\{w_i, z_j, \eta_k, \theta_l\}$ be a *relative coordinate system*, that is, a set of sections of $\mathcal{B}_X$, with the w_i, z_j's even and the η_k, θ_l's odd, such that $\{w_i, \eta_k\}$ are coordinates on $(Y, \mathcal{B}_Y)$ and $\{z_j, \theta_l\}$ are coordinates on the fibers $(X_y, \mathcal{B}_{X_y})$. One has bases $\{\partial/\partial z_j, \partial/\partial\theta_l\}$ of $\mathcal{D}er\,(\mathcal{B}_X/\mathcal{B}_Y)$ and $\{dz_j, d\theta_l\}$ of $\Omega^1_{\mathcal{B}_X/\mathcal{B}_Y}$ (these can be considered also as bases for $\mathcal{D}er\,\mathcal{B}_{X_y}$ and $\Omega^1_{\mathcal{B}_{X_y}}$).

Definition 2.4. The relative Berezinian sheaf of the family $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is the Berezinian sheaf $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ associated with the locally free $\mathcal{B}_X$-module $\Omega^1_{\mathcal{B}_X/\mathcal{B}_Y}$.

Up to a change of parity, one can describe $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ as the cohomology group $H(K^\bullet)$ of the Koszul complex $K^\bullet = S^\bullet_{\mathcal{B}_X}(\Pi\Omega^1_{\mathcal{B}_X/\mathcal{B}_Y} \oplus (\Omega^1_{\mathcal{B}_X/\mathcal{B}_Y})^\vee)$ [14], where Π is the parity change functor, and $^\vee$ denotes the dual module; in this sense one denotes by $[dz_1 \cdots dz_n \otimes \partial/\partial\theta_1 \cdots \partial/\partial\theta_{n'}]$ the local basis of $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ corresponding to a relative coordinate system $\{w_i, z_j, \eta_k, \theta_l\}$ [as a matter of fact, $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} \simeq \Pi^n H(K^\bullet)$]. Thus, if $\{w'_i, z'_j, \eta'_k, \theta'_l\}$ is another relative coordinate system, one has

$$\left[dz'_1 \cdots dz'_n \otimes \frac{\partial}{\partial\theta'_1} \cdots \frac{\partial}{\partial\theta'_{n'}}\right] = \left[dz_1 \cdots dz_n \otimes \frac{\partial}{\partial\theta_1} \cdots \frac{\partial}{\partial\theta_{n'}}\right] \mathrm{Ber}\, J\,, \qquad (2.2)$$

where J is the Jacobian matrix of the coordinate transformation, and Ber is the Berezin determinant.

For an oriented family of real graded manifolds $\pi : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ of relative dimension (m, m'), the sections of the corresponding Berezinian sheaf $\mathrm{Ber}_{\mathcal{A}_X/\mathcal{A}_Y}$ (defined as in the complex analytic case) can be interpreted in a natural way as "graded volume forms" on the family, and one has a *fiberwise Berezin integration*

$$\int_{\mathcal{A}_X/\mathcal{A}_Y} : \pi_* \mathrm{Ber}_{\mathcal{A}_X/\mathcal{A}_Y} \to \mathcal{A}_Y\,, \qquad (2.3)$$

which is defined as follows (cf. ref. [11], and also ref. [4] for the absolute case).

Definition 2.5.

(1) Let $\{y_i, x_j, \mu_k, \nu_l\}$ be a relative coordinate system, and let ω be a section of $\pi_* \mathrm{Ber}_{\mathcal{A}_X/\mathcal{A}_Y}$ such that

$$\omega = \left[dx_1 \cdots dx_m \otimes \frac{\partial}{\partial\nu_1} \cdots \frac{\partial}{\partial\nu_{m'}}\right] \sum_{\alpha,\beta} f_{\alpha\beta}(y, x)\mu_\alpha\nu_\beta\,,$$

where the $f_{\alpha\beta}$'s are sections of $\mathcal{C}_X$, α, β are multi-indices of the type $\beta = \{\beta_1, \ldots, \beta_s\}$ with $1 \le \beta_1 < \cdots < \beta_s \le m'$, and $\nu_\beta = \nu_{\beta_1} \wedge \cdots \wedge \nu_{\beta_s}$. Then one defines

$$\int_{\mathcal{A}_X/\mathcal{A}_Y} \omega = \sum_\alpha \left(\int_{X/Y} f_{\alpha\varpi}\, dx_1 \cdots dx_m \right) \mu_\alpha\,,$$

where ϖ is the multi-index $\{1, 2, \ldots, m'\}$, and $\int_{X/Y}$ is the fiberwise integration [5] of sections of the sheaf $\pi_*\Omega^m_{X/Y}$.

(2) For a generic section ω, one defines $\int_{\mathcal{A}_X/\mathcal{A}_Y}$ by additivity, using a partition of unity and (1).

2.3. RELATIVE SERRE DUALITY

Let $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ be a family of complex graded manifolds of relative dimension (n, n'). In this context, the Berezinian sheaf $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$, analogously to the sheaf $\kappa_{X/Y}$ of relative holomorphic n-forms in the non-graded case, plays the role of *dualizing sheaf*; that is, if $\mathcal{M}$ is a coherent $\mathcal{B}_X$-module, there is a canonical isomorphism of $\mathcal{B}_Y$-modules [15,16,20]

$$R\pi_* R\mathcal{H}om_{\mathcal{B}_X}(\mathcal{M}, \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}(-n)) \simeq R\mathcal{H}om_{\mathcal{B}_Y}(R\pi_*\mathcal{M}, \mathcal{B}_Y). \tag{2.4}$$

This assertion can be proved as in the non-graded case [8], and one obtains in particular (cf. ref. [5])

$$R^n\pi_* \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} \simeq (\pi_*\mathcal{B}_X)^\vee. \tag{2.5}$$

2.4. THE PICARD GROUP OF A GRADED FAMILY

A *line bundle* over a complex graded manifold $(X, \mathcal{B}_X)$ is a locally free $\mathcal{B}_X$-module either of rank $(1,0)$ or $(0,1)$; the parity change functor Π establishes a one-to-one correspondence between the groups of isomorphism classes of line bundles of fixed rank. The *Picard group* of $(X, \mathcal{B}_X)$ is the group $\mathrm{Pic}(X, \mathcal{B}_X)$ of isomorphism classes of line bundles, disregarding parity.

On the analogy of the non-graded case, the elements in $\mathrm{Pic}(X, \mathcal{B}_X)$ are characterized by Čech one-cocycles of the sheaf $(\mathcal{B}_X)_0^*$ (invertible and even sections of $\mathcal{B}_X$), called *transition functions*, and there is an isomorphism $\mathrm{Pic}(X, \mathcal{B}_X) \simeq H^1(X, (\mathcal{B}_X)_0^*)$. If $(X, \mathcal{O}_X)$ is the complex analytic manifold underlying $(X, \mathcal{B}_X)$, the structural projection $\mathcal{B}_X \to \mathcal{O}_X$ induces a morphism

$$\mathrm{Pic}(X, \mathcal{B}_X) \to \mathrm{Pic}(X), \quad \mathcal{L} \mapsto \mathcal{L} \otimes_{\mathcal{B}_X} \mathcal{O}_X \equiv \tilde{\mathcal{L}}, \tag{2.6}$$

which has, in general, a kernel and a cokernel. In the particular case that $(\mathcal{B}_X)_0^* \simeq \mathcal{O}_X^*$, (2.6) is of course an isomorphism [for instance, this happens if the odd dimension of $(X, \mathcal{B}_X)$ is 1].

One can, however, identify the *Chern classes* $c_1(\mathcal{L}) = c_1(\tilde{\mathcal{L}}) \in H^2(X, \mathbb{Z})$, where $c_1(\mathcal{L})$ is defined as minus the image of $\mathcal{L} \in \mathrm{Pic}(X, \mathcal{B}_X)$, via the cohomology morphism induced by the first line of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & (\mathcal{B}_X)_0 & \xrightarrow{\exp 2\pi i} & (\mathcal{B}_X)_0^* & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_X & \xrightarrow{\exp 2\pi i} & \mathcal{O}_X^* & \longrightarrow & 0 \end{array} ; \tag{2.7}$$

the desired identification is then provided by the cohomology diagram induced by (2.7).

The concept of Picard group and related notions in the graded setting can be extended to the context of families, exactly as happens in the non-graded case

[5]. The *relative Picard sheaf* of the family $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is the sheaf $R^1\pi_*(\mathcal{B}_X)_0^*$; the *(restricted) relative Picard group* is the space $\mathrm{Pic}(\mathcal{B}_X/\mathcal{B}_Y) = \Gamma(Y, R^1\pi_*(\mathcal{B}_X)_0^*)$, and there is a functorial morphism $\phi : \mathrm{Pic}(X, \mathcal{B}_X) \to \mathrm{Pic}(\mathcal{B}_X/\mathcal{B}_Y)$, $\mathcal{L} \mapsto [\mathcal{L}]$. The *relative Chern class* c_1 is minus the morphism $R^1\pi_*(\mathcal{B}_X)_0^* \to R^2\pi_*\mathbb{Z}$ induced by (2.7) (if the underlying complex family is a family of Riemann surfaces, the Chern class can be regarded as a section of $R^2\pi_*\mathbb{C}$); one denotes in the same way the corresponding morphism between spaces of global sections, and one proves that c_1 is functorial and commutes with ϕ as in the non-graded case. A section of $R^1\pi_*(\mathcal{B}_X)_0^*$ is said to be *flat* if it lies in the image of the sheaf morphism $R^1\pi_*\pi^{-1}(\mathcal{B}_Y)_0^* \to R^1\pi_*(\mathcal{B}_X)_0^*$ induced by the natural morphism $\pi^{-1}(\mathcal{B}_Y)_0^* \hookrightarrow (\mathcal{B}_X)_0^*$.

3. SUSY-curves

3.1. BASIC DEFINITIONS

Let us recall a brief motivation, in terms of conformal structures, of why a "super Riemann surface" is not simply a (1,1) dimensional complex graded manifold. In the ordinary case, given a two-dimensional manifold X, the complex structures on X are in a one-to-one correspondence with the conformal classes of Riemannian structures on X, and conformal changes of the metric structure correspond to holomorphic maps, i.e., they map the vector field $\partial/\partial z$ into a multiple of itself. One might wonder what is the analog of this situation in the graded case. If $(X, \mathcal{B}_X)$ is a complex graded manifold of dimension (1,1), with local coordinates (z, θ), locally the even vector field $\partial/\partial z$ has an odd "square root": if $D = \partial/\partial\theta + \theta\partial/\partial z$, then $D^2 = \partial/\partial z$. We say that $(X, \mathcal{B}_X)$ is endowed with a conformal structure if the local D's define a global rank (0,1) submodule of the tangent sheaf (one can equivalently require the existence of a maximally non-integrable rank (0,1) submodule of the tangent sheaf, and then proves that that submodule is locally generated by D). Thus, in the graded case a conformal structure is richer than a complex structure, and it comes out that in order to extend to the graded setting several classical constructions one needs indeed a conformal structure.

Definition 3.1. A graded Riemann surface, or SUSY-curve [13], is a family of complex graded manifolds $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ of relative dimension (1, 1), endowed with a locally free rank (0,1) subsheaf $\mathcal{D}$ of $\mathcal{D}er\,(\mathcal{B}_X/\mathcal{B}_Y)$, such that the morphism

$$\mathcal{D} \otimes_{\mathcal{B}_X} \mathcal{D} \xrightarrow{[\ ,\ ]\ \mathrm{mod}\,\mathcal{D}} \mathcal{D}er\,(\mathcal{B}_X/\mathcal{B}_Y)/\mathcal{D} \qquad (3.1)$$

is an isomorphism of graded $\mathcal{B}_X$-modules.

Here [,] is the graded Lie bracket. The subsheaf $\mathcal{D}$ is called a *conformal* *structure*; we shall briefly denote a SUSY-curve by $(X/Y,\mathcal{D})$. For any $y \in Y$, the subsheaf $\mathcal{D}_y = \mathcal{D}|_{X_y} \otimes_{(\mathcal{B}_Y)_y} k(y)$ of $\mathcal{D}er\,\mathcal{B}_{X_y}$ endows the fiber $(X_y, \mathcal{B}_{X_y})$ of a SUSY-curve $(X/Y,\mathcal{D})$ over y with the structure of a SUSY-curve (as a family over the point y), which we denote briefly by $(X_y, \mathcal{D}_y)$.

We would like to recall some results about SUSY-curves, which relate definition 3.1 to the more intuitive notions of SUSY-curve as suggested by analogy with the non-graded case or by physical motivations (proofs can be found in refs. [10,12,13]).

Lemma 3.2. *Any SUSY-curve $(X/Y,\mathcal{D})$ admits relative coordinate systems $\{w_i, z, \eta_k, \theta\}$, called conformal, such that $\mathcal{D}$ is locally generated by the odd derivation*

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}. \quad \square \tag{3.2}$$

We shall denote conformal coordinates by (z,θ), and call *conformal* the coordinate changes which preserve conformal coordinates. A simple computation shows that conformal changes have the form

$$z' = \varphi(z) \pm \theta\psi(z)\sqrt{\frac{\partial\varphi}{\partial z} + \psi(z)\frac{\partial\psi}{\partial z}} = \varphi(z) \pm \theta\psi(z)\sqrt{\frac{\partial\varphi}{\partial z}},$$

$$\theta' = \psi(z) \pm \theta\sqrt{\frac{\partial\varphi}{\partial z} + \psi(z)\frac{\partial\psi}{\partial z}},$$

where φ (ψ) is an even (odd) holomorphic graded function which additionally depends on the coordinates of $(Y, \mathcal{B}_Y)$.

If the "parameter space" $(Y, \mathcal{B}_Y)$ has no odd parameters (i.e., it is an ordinary manifold), then the SUSY-curve is *split*, namely, the structure sheaf of the total space is an exterior algebra.

Lemma 3.3. *Let $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{O}_Y)$ be a family of complex analytic graded manifolds of relative dimension $(1,1)$, with $(Y, \mathcal{O}_Y)$ an ordinary complex manifold. Then:*

(i) $\mathcal{B}_X \simeq \Lambda^\bullet_{\mathcal{O}_X} L$, where L is a line bundle on $(X, \mathcal{O}_X)$;

(ii) the specification of a conformal structure $\mathcal{D}$ on the family is equivalent to L being a "relative spin structure" on $(X, \mathcal{O}_X)$, that is, to an isomorphism $L \otimes_{\mathcal{O}_X} L \simeq \kappa_{X/Y}$, where $\kappa_{X/Y}$ is the sheaf of relative holomorphic one-differentials on $(X, \mathcal{O}_X)$; moreover, one has $\mathcal{D}^* \simeq L \otimes_{\mathcal{O}_X} \Pi\mathcal{B}_X$. $\square$

Notice that the particular case of a SUSY-curve over a point (for instance, the fiber of a generic SUSY-curve) satisfies the hypotheses of this lemma.

Proposition 3.4. *If $(X/Y,\mathcal{D})$ is a SUSY-curve, there is a canonical isomorphism of $\mathcal{B}_X$-modules*

$$\mathcal{D}^\vee \xrightarrow{\sim} \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}\,. \tag{3.3}$$

Proof. In conformal coordinates (z,θ), the module $\mathcal{D}^\vee$ has a local basis $[d\theta]$ given by the image of $d\theta \in \Omega^1_{\mathcal{B}_X/\mathcal{B}_Y}$ under the epimorphism $\Omega^1_{\mathcal{B}_X/\mathcal{B}_Y} \to \mathcal{D}^\vee$ (dual of the natural inclusion), and $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ has a basis $[dz \otimes \partial/\partial\theta]$. One defines the isomorphism (3.3) locally by letting $[d\theta] \mapsto [dz \otimes \partial/\partial\theta]$, and proves that it does not depend on the choice of the conformal coordinates. □

3.2. BEREZINIAN CALCULUS

If $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is a family of complex graded manifolds of relative dimension $(1,1)$, one can define, analogously to the non-graded case, the *sheaves of smooth relative* (p,q)*-forms*,

$$\Omega^{p,q} = (\mathcal{A}_X \otimes_{\mathcal{B}_X} (\Omega^1_{\mathcal{B}_X/\mathcal{B}_Y})^p) \otimes_{\overline{\mathcal{B}}_X} (\overline{\Omega}^1_{\mathcal{B}_X/\mathcal{B}_Y})^q , \qquad p,q = 0,1, \tag{3.4}$$

where $(X, \mathcal{A}_X)$ is the graded smooth manifold underlying $(X, \mathcal{B}_X)$, and a bar denotes complex conjugation. Moreover, $\Omega^{1,0} \oplus \Omega^{0,1} \simeq \Omega_{\mathcal{A}_X/\mathcal{A}_Y}$, and the graded relative differential induces differential operators

$$\partial_r : \mathcal{A}_X \to \Omega^{1,0}, \qquad \overline{\partial}_r : \mathcal{A}_X \to \Omega^{0,1} . \tag{3.5}$$

However, relative (p,q)-forms are not the correct forms to construct a differential calculus, because there are no top-degree forms, and they are not related to "fiberwise integrable objects". Therefore, it is convenient to define a new kind of forms in terms of Berezinian sheaves; such forms are naturally defined, and exist only, on SUSY-curves.

Definition 3.5. Let $(X/Y, \mathcal{D})$ be a SUSY-curve, $d_r : \mathcal{B}_X \to \Omega^1_{\mathcal{B}_X/\mathcal{B}_Y}$ the (graded) relative differential, and $\varrho : \Omega^1_{\mathcal{B}_X/\mathcal{B}_Y} \to \mathcal{D}^\vee \simeq \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ the natural epimorphism. The holomorphic Berezin differential $\hat{\partial}$ is the $\pi^{-1}\mathcal{B}_Y$-linear morphism

$$\hat{\partial} = \varrho \circ d_r : \mathcal{B}_X \to \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} . \tag{3.6}$$

In a conformal coordinate system (z,θ) one has $\hat{\partial} f = [dz \otimes \partial/\partial\theta] D(f)$, with D as in (3.2).

Lemma 3.6. *There is an exact sequence of* $\pi^{-1}\mathcal{B}_Y$*-modules*

$$0 \longrightarrow \pi^{-1}\mathcal{B}_Y \longrightarrow \mathcal{B}_X \xrightarrow{\hat{\partial}} \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} \longrightarrow 0 . \tag{3.7}$$

Proof. It is enough to prove exactness on the stalks, which permits us to use conformal coordinates (z,θ). Trivially, a section f in $\pi^{-1}\mathcal{B}_Y$ satisfies $\hat{\partial} f = 0$; furthermore, $0 = \hat{\partial} f = [dz \otimes \partial/\partial\theta] D(f)$, with $f = f_0(z) + \theta f_1(z)$, implies that $\partial f_0/\partial z = 0$ and $f_1 = 0$, so that f is a section of $\pi^{-1}\mathcal{B}_Y$. To conclude, one verifies that the section $f = \int g_1(z)\, dz + \theta g_0(z)$ of $\mathcal{B}_X$ is a pre-image of the section $[dz \otimes \partial/\partial\theta](g_0(z) + \theta g_1(z))$ of $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ under $\hat{\partial}$ (where $\int \mathrm{d}z$ is the indefinite path integral). □

Since one also has the exact sequence of $\pi^{-1}\mathcal{B}_Y$-modules

$$0 \longrightarrow \pi^{-1}\mathcal{B}_Y \longrightarrow \mathcal{B}_X \xrightarrow{d_r} \mathcal{Z}^1_{\mathcal{B}_X/\mathcal{B}_Y} \longrightarrow 0,$$

one can identify the sheaves $\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ and $\mathcal{Z}^1_{\mathcal{B}_X/\mathcal{B}_Y}$ as $\pi^{-1}\mathcal{B}_Y$-modules; however, this isomorphism shall be regarded in a sense as accidental, and will not be exploited in the sequel.

Definition 3.7. We define the sheaves of Berezin (p,q)-forms as the sheaves over X

$$\mathcal{A}^{p,q} = (\mathcal{A}_X \otimes_{\mathcal{B}_X} (\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y})^p) \otimes_{\overline{\mathcal{B}}_X} (\overline{\mathrm{Ber}}_{\mathcal{B}_X/\mathcal{B}_Y})^q, \quad p,q = 0,1. \tag{3.8}$$

Sections of these sheaves have elsewhere been called $(p/2, q/2)$-differentials [6].

Lemma 3.8. *There is a canonical isomorphism of $\mathcal{A}_X$-modules*

$$\mathcal{A}^{1,1} = \mathcal{A}^{1,0} \otimes_{\mathcal{A}_X} \mathcal{A}^{0,1} \xrightarrow{\sim} \mathrm{Ber}_{\mathcal{A}_X/\mathcal{A}_Y}.$$

Proof. In conformal coordinates (z,θ) the $\mathcal{A}_X$-module $\mathcal{A}^{1,1}$ has a basis $[dz \otimes \partial/\partial\theta] \otimes [d\bar{z} \otimes \partial/\partial\bar{\theta}]$, and $\mathrm{Ber}_{\mathcal{A}_X/\mathcal{A}_Y}$ has a basis $[dz\, d\bar{z} \otimes \partial/\partial\theta\, \partial/\partial\bar{\theta}]$. One defines locally this isomorphism by letting $[dz \otimes \partial/\partial\theta] \otimes [d\bar{z} \otimes \partial/\partial\bar{\theta}] \mapsto [dz\, d\bar{z} \otimes \partial/\partial\theta\, \partial/\partial\bar{\theta}]$, and then proves that it is independent of the choice of conformal coordinates. □

The epimorphism $\varrho : \Omega^1_{\mathcal{B}_X/\mathcal{B}_Y} \to \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$ induces epimorphisms $\varrho^{p,q} : \Omega^{p,q} \to \mathcal{A}^{p,q}$; composing with the differential operators ∂_r and $\overline{\partial}_r$ [cf. (3.5)] one has:

Definition 3.9. The morphisms

$$\delta = \varrho^{1,0} \circ \partial_r : \mathcal{A}_X \to \mathcal{A}^{1,0}, \qquad \overline{\delta} = \varrho^{0,1} \circ \overline{\partial}_r : \mathcal{A}_X \to \mathcal{A}^{0,1}$$

are the smooth Berezin differentials.

The differential δ is $\overline{\mathcal{B}}_X$-linear, while $\overline{\delta}$ is $\mathcal{B}_X$-linear. In conformal coordinates (z,θ) the expression of the smooth Berezin differentials is

$$\delta f = [dz \otimes \partial/\partial\theta] D(f), \qquad \overline{\delta} f = [d\bar{z} \otimes \partial/\partial\bar{\theta}] \overline{D}(f).$$

Proposition 3.10. *There are exact sequences of $\mathcal{B}_X$-modules and $\overline{\mathcal{B}}_X$-modules, respectively,*

$$0 \longrightarrow \mathcal{B}_X \longrightarrow \mathcal{A}_X \xrightarrow{\overline{\delta}} \mathcal{A}^{0,1} \longrightarrow 0, \qquad 0 \longrightarrow \overline{\mathcal{B}}_X \longrightarrow \mathcal{A}_X \xrightarrow{\delta} \mathcal{A}^{1,0} \longrightarrow 0.$$

Proof. One proves the exactness of the first sequence (the second is analogous) in the stalks, which allows us to use conformal coordinates (z,θ). If f is a section in $\mathcal{B}_X$, in the induced coordinates $(z,\bar{z},\theta,\overline{\theta})$ in $\mathcal{A}_X$ it does not depend on $\bar{z}$ and $\overline{\theta}$, so that $\overline{\delta}f = 0$; moreover, $0 = \overline{\delta}f = [d\bar{z}\otimes\partial/\partial\overline{\theta}]\overline{D}(f)$, with $f = f_0(z,\bar{z}) + \theta f_1(z,\bar{z}) + \overline{\theta}f_2(z,\bar{z}) + \theta\overline{\theta}f_3(z,\bar{z})$, implies $\partial f_0/\partial\bar{z} = \partial f_2/\partial\bar{z} = 0$ and $f_2 = f_3 = 0$, so that f is a section of $\mathcal{B}_X$. Finally, $\overline{\delta}$ is surjective because it is the composition of surjective morphisms. □

We extend the smooth Berezin differentials to morphisms $\delta : \mathcal{A}^{0,1} \to \mathcal{A}^{1,1}$ and $\overline{\delta} : \mathcal{A}^{1,0} \to \mathcal{A}^{1,1}$ by letting $\delta(f\otimes\bar{\omega}) = \delta f\otimes\bar{\omega}$ and $\overline{\delta}(f\otimes\omega) = \overline{\delta}f\otimes\omega$. In conformal coordinates (z,θ), this implies $\delta([d\bar{z}\otimes\partial/\partial\overline{\theta}]g) = -[dz\,d\bar{z}\otimes\partial/\partial\theta\,\partial/\partial\overline{\theta}]D(g)$, and $\overline{\delta}([dz\otimes\partial/\partial\theta]g) = -[dz\,d\bar{z}\otimes\partial/\partial\theta\,\partial/\partial\overline{\theta}]\overline{D}(g)$.

Thus, the smooth Berezin differentials act as sheaf morphisms

$$\delta : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}, \quad p = 0,\, q = 0,1,$$
$$\overline{\delta} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}, \quad p = 0,1,\, q = 0.$$

Lemma 3.11. *The identity* $\delta\circ\overline{\delta} + \overline{\delta}\circ\delta = 0$ *holds.*

Proof. The claim reduces to proving the easy identity $D\circ\overline{D} + \overline{D}\circ D = 0$. □

Proposition 3.12. *The following sequences of* $\mathcal{B}_X$*-modules and* $\overline{\mathcal{B}}_X$*-modules, respectively, are exact:*

$$0 \longrightarrow \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} \longrightarrow \mathcal{A}^{1,0} \xrightarrow{\overline{\delta}} \mathcal{A}^{1,1} \longrightarrow 0,$$
$$0 \longrightarrow \overline{\mathrm{Ber}}_{\mathcal{B}_X/\mathcal{B}_Y} \longrightarrow \mathcal{A}^{0,1} \xrightarrow{\delta} \mathcal{A}^{1,1} \longrightarrow 0.$$

Proof. One tensorizes the exact sequences in proposition 3.10 by $\otimes_{\overline{\mathcal{B}}_X}\overline{\mathrm{Ber}}_{\mathcal{B}_X/\mathcal{B}_Y}$ (or $\otimes_{\mathcal{B}_X}\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y}$). □

Definition 3.13. The (relative) de Rham–Berezin sheaf $DRB_{X/Y}$ of the SUSY-curve $(X/Y,\mathcal{D})$ is the sheaf over Y

$$DRB_{X/Y} \equiv \frac{\pi_*\mathcal{A}^{1,1}}{\overline{\delta}\pi_*\mathcal{A}^{1,0}\oplus\delta\pi_*\mathcal{A}^{0,1}}.$$

The relationship between the relative de Rham sheaf and the de Rham–Berezin sheaf can be established by constructing a morphism

$$\Psi : \mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y} \to \mathcal{A}^{1,1} \tag{3.9}$$

as follows: a section ξ of $\mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y}$ is locally $\xi = d_r\omega$ for a section ω of $\Omega^1_{\mathcal{A}_X/\mathcal{A}_Y}$ (Poincaré lemma); then, if $\omega = \omega^{1,0} + \omega^{0,1}$ with $\omega^{p,q}$ a section of $\Omega^{p,q}$, one defines locally Ψ by the formula

$$\Psi(\xi) = \overline{\delta}(\varrho^{1,0}(\omega^{1,0})) + \delta(\varrho^{0,1}(\omega^{0,1}))\,.$$

This definition is in fact global, because, if $\xi = d_r\omega'$, with $\omega' = \omega + d_r f$ and f a section of $\mathcal{A}_X$, one has $\overline{\delta}(\varrho^{1,0}((d_r f)^{1,0})) + \delta(\varrho^{0,1}((d_r f)^{0,1})) = 0$.

Composing the morphism $\pi_*\mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y} \to \pi_*\mathcal{A}^{1,1}$ induced by (3.9) with the projection onto the quotient $DRB_{X/Y}$, one obtains a sheaf morphism $\pi_*\mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y} \to DRB_{X/Y}$; an easy computation shows that it factorizes through $DR^2_{\mathcal{A}_X/\mathcal{A}_Y}$; taking global sections, one has a commutative diagram of $\mathbb{C}$-modules,

$$\begin{array}{ccc} \Gamma(X, \mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y}) & \longrightarrow & \Gamma(X, \mathcal{A}^{1,1}) \\ {\scriptstyle p}\downarrow & & \downarrow \\ \Gamma(Y, DR^2_{\mathcal{A}_X/\mathcal{A}_Y}) & \longrightarrow & \Gamma(Y, DRB_{X/Y}) \end{array} \quad . \tag{3.10}$$

Proposition 3.14 (Stokes theorem). *Let τ and $\bar{\tau}'$ be sections of $\pi_*\mathcal{A}^{1,0}$ and $\pi_*\mathcal{A}^{0,1}$ respectively. Then*

$$\int_{\mathcal{A}_X/\mathcal{A}_Y} \overline{\delta}\tau = \int_{\mathcal{A}_X/\mathcal{A}_Y} \delta\bar{\tau}' = 0\,.$$

Thus, fiberwise Berezin integration induces a morphism

$$\int_{\mathcal{A}_X/\mathcal{A}_Y} : DRB_{X/Y} \to \mathcal{A}_Y\,.$$

Proof. It is enough to prove this result when, after choosing conformal coordinates (z, θ), one has $\tau = [dz \otimes \partial/\partial\theta] f$ and $\bar{\tau}' = [d\bar{z} \otimes \partial/\partial\bar{\theta}] g$, with f, g sections of $\mathcal{A}_X$; then $\int_{\mathcal{A}_X/\mathcal{A}_Y} \overline{\delta}\tau = \int_{X/Y} \overline{D}(f)_{\theta\bar{\theta}}\, \mathrm{d}z\, \mathrm{d}\bar{z}$ and $\int_{\mathcal{A}_X/\mathcal{A}_Y} \delta\bar{\tau}' = \int_{X/Y} D(g)_{\theta\bar{\theta}}\, \mathrm{d}z\, \mathrm{d}\bar{z}$. One verifies that $\overline{D}(f)_{\theta\bar{\theta}}\, dz \wedge d\bar{z}$ and $D(g)_{\theta\bar{\theta}}\, dz \wedge d\bar{z}$ are ordinary relative two-forms, exact with respect to d_r, and one proves the first assertion from Stokes' theorem for $\int_{X/Y}$. This yields a morphism from the quotient presheaf $\pi_*\mathcal{A}^{1,1}/(\overline{\delta}\pi_*\mathcal{A}^{1,0} \oplus \delta\pi_*\mathcal{A}^{0,1})$ into $\mathcal{A}_Y$, and one concludes by factorizing through the associated sheaf $DRB_{X/Y}$. □

4. Line bundles over SUSY-curves

4.1. FLATNESS THEOREM

Let $(X/Y, \mathcal{D})$ be a SUSY-curve. In order to derive a characterization of the flat sections of the relative Picard sheaf $R^1\pi_*(\mathcal{B}_X)_0^*$, which parallels the analogous result in the non-graded case, let us consider the commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ & & & & \mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{B}_X & \longrightarrow & \mathcal{B}_X^* & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi^{-1}\mathcal{B}_Y & \longrightarrow & \pi^{-1}\mathcal{B}_Y^* & \longrightarrow & 0 \end{array} \quad . \tag{4.1}$$

Applying the higher direct image functor, one has $R^2\pi^*\mathcal{B}_X = 0$, so that from the central vertical sequence one obtains an epimorphism

$$\alpha : R^1\pi_* \,\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} \to R^2\pi_*\pi^{-1}\mathcal{B}_Y\,, \tag{4.2}$$

and from the bottom horizontal sequence a morphism

$$\beta : R^2\pi_*\mathbb{Z} \to R^2\pi_*\pi^{-1}\mathcal{B}_Y\,.$$

Lemma 4.1. *The morphism β is injective.*

Proof. After taking derived functors in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & = & \mathbb{Z} \\ \downarrow & & \downarrow \\ \pi^{-1}\mathcal{B}_Y & \longrightarrow & \pi^{-1}\mathcal{O}_Y \end{array}$$

the claim follows from the analogous result in the non-graded case ([5], lemma 3.10). $\square$

From (4.1) one has therefore the commutative diagram

$$
\begin{array}{ccccccc}
0 & & & & & & \\
\uparrow & & & & & & \\
R^2\pi_*\pi^{-1}\mathcal{B}_Y & & & & & & \\
{\scriptstyle\alpha}\uparrow & & & & & & \\
R^1\pi_*\,\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} & & & & & & \\
\uparrow & & & & & & \\
R^1\pi_*\mathcal{B}_X & \longrightarrow & R^1\pi_*\mathcal{B}_X^* & \xrightarrow{c_1} & R^2\pi_*\mathbb{Z} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & & & \\
R^1\pi_*\pi^{-1}\mathcal{B}_Y & \longrightarrow & R^1\pi_*\pi^{-1}\mathcal{B}_Y^* & \longrightarrow & 0 & &
\end{array}
$$

Restricting this diagram to the even parts of the modules, one can deduce by simple diagram chasing that any flat section of the relative Picard sheaf $R^1\pi_*(\mathcal{B}_X)_0^*$ has a vanishing relative Chern class.

In order to prove the converse, one should demonstrate that the epimorphism $\alpha_0 : R^1\pi_*(\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y})_0 \to R^2\pi_*\pi^{-1}(\mathcal{B}_Y)_0$ is bijective. However, this is false in general; it is possible for instance to construct SUSY-curves having the following properties:

(i) the base space Y is a point, so that the derived functors reduce to cohomology groups;

(ii) $\dim H^1(X, (\mathrm{Ber}_{\mathcal{B}_X/\mathcal{B}_Y})_0) = 2$, while $\dim H^2(X, \pi^{-1}(\mathcal{B}_Y)_0) = 1$.

Such an example can be found in ref. [6], page 294.

A condition assuring that α is an isomorphism is the analog of the universal connectedness that we have assumed in the ordinary case, namely, $\pi_*\mathcal{B}_X \simeq \mathcal{B}_Y$.[#2] The proof that under this condition α is in fact an isomorphism goes as in the ordinary case (cf. ref. [5], lemma 3.9).

Thus, we arrive at the following result.

Theorem 4.2. *Let $(X/Y, \mathcal{D})$ be a SUSY-curve, and assume that $\pi_*\mathcal{B}_X \simeq \mathcal{B}_Y$. Then, any $y \in Y$ has an open neighborhood V such that a section of $R^1\pi_*(\mathcal{B}_X)_0^*$ has a vanishing relative Chern class if and only if it is flat.* □

Remark 4.3. The fact that the morphism α_0 is not bijective implies that an analog of Grauert's cohomology base change theorem does not hold for SUSY-curves.

[#2] One should notice that the moduli space of SUSY-curves does not fulfill such a requirement.

Indeed, if such a theorem held, one would have

$$R^1\pi_* \operatorname{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} \otimes_{\mathcal{B}_Y} k(y) \simeq H^1(X_y, \operatorname{Ber}_{\mathcal{B}_{X_y}}) .$$

Since $\operatorname{Ber}_{\mathcal{B}_{X_y}} \simeq \kappa_{X_y} \oplus \kappa_{X_y}^{1/2}$, with $\kappa_{X_y}^{1/2}$ a spin structure over the fiber X_y, one has $R^1\pi_* \operatorname{Ber}_{\mathcal{B}_X/\mathcal{B}_Y} \otimes_{\mathcal{B}_Y} k(y) \simeq \mathbb{C} \oplus \mathbb{C}^q$, where either $q = 0$ or $q = 1$ for a generic position in the moduli space. On the other hand, $R^2\pi_*\pi^{-1}\mathcal{B}_Y \otimes_{\mathcal{B}_Y} k(y) \simeq (\mathcal{B}_Y)_y \otimes_{(\mathcal{B}_Y)_y} k(y) \simeq \mathbb{C}$. Restricting to even parts, the Nakayama lemma would imply that α is bijective.

4.2. CONFORMAL CONNECTIONS AND GAUSS–BONNET THEOREM

Let $(X, \mathcal{B}_X)$ be a complex graded manifold, and $\mathcal{L}$ a line bundle over it. As in the ordinary case, one can prove that $c_1(\mathcal{L}) = (\mathrm{i}/2\pi)[K]$, where $[K]$ is the de Rham cohomology class of a (smooth) curvature form K on $\mathcal{L}$. A procedure analogous to that employed in ref. [5] for families of complex manifolds allows us to define, for a family of complex analytic graded manifolds $\pi : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$, a *(smooth) relative graded connection* over $\mathcal{L}$ as a morphism of $\mathbb{C}$-modules $\nabla_r : \mathcal{L} \to \Omega^1_{\mathcal{A}_X/\mathcal{A}_Y} \otimes_{\mathcal{B}_X} \mathcal{L}$ which satisfies the Leibniz rule $\nabla_r(f\sigma) = d_r f \otimes \sigma + f\nabla_r(\sigma)$. The *curvature* K_r of a relative graded connection ∇_r is the $\mathbb{C}$-module morphism $K_r = \nabla_r^2 : \mathcal{L} \to \Omega^2_{\mathcal{A}_X/\mathcal{A}_Y} \otimes_{\mathcal{B}_X} \mathcal{L}$, which is $\mathcal{B}_X$-linear and determines a global section K_r of $\Omega^2_{\mathcal{A}_X/\mathcal{A}_Y}$; this is closed under the relative differential, that is, $K_r \in \Gamma(X, \mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y})$, and its projection $[K_r] \in \Gamma(Y, DR^2_{\mathcal{A}_X/\mathcal{A}_Y})$ does not depend on the relative connection over $\mathcal{L}$. If $[\mathcal{L}]$ denotes the image of $\mathcal{L}$ in $\operatorname{Pic}(\mathcal{B}_X/\mathcal{B}_Y)$, one has an identification (relative Gauss–Bonnet theorem)

$$c_1([\mathcal{L}]) = (\mathrm{i}/2\pi)[K_r] \in \Gamma(Y, DR^2_{\mathcal{A}_X/\mathcal{A}_Y}) , \tag{4.2}$$

where $c_1[\mathcal{L}] \in \Gamma(Y, R^2\pi_*\mathbb{C})$ is regarded as a section in $\Gamma(Y, R^2\pi_*\pi_{-1}\mathcal{A}_Y) \simeq \Gamma(Y, DR^2_{\mathcal{A}_X/\mathcal{A}_Y})$.

When the family under consideration is a SUSY-curve, it is possible to state this result in terms of fiberwise Berezin integration, analogously to the non-graded case. An added difficulty in this context is that the elements in $\Gamma(Y, DR^2_{\mathcal{A}_X/\mathcal{A}_Y})$ are not fiberwise integrable, so that it is necessary to define a new kind of connections, whose curvature forms are Berezin forms.

Definition 4.4. Given a SUSY-curve $(X/Y, \mathcal{D})$, let $\mathcal{A}^{p,q}$ $(p, q = 0, 1)$ be the sheaves of Berezin (p, q) forms, and let $\mathcal{L}$ be a line bundle over $(X, \mathcal{B}_X)$. A conformal connection over $\mathcal{L}$ is a pair $(\hat{\nabla}, \hat{\overline{\nabla}})$ of even morphisms of graded $\pi^{-1}\mathcal{B}_Y$-modules

$$\hat{\nabla} : \mathcal{L} \to \mathcal{A}^{1,0} \otimes_{\mathcal{B}_X} \mathcal{L}, \qquad \hat{\overline{\nabla}} : \mathcal{L} \to \mathcal{A}^{0,1} \otimes_{\mathcal{B}_X} \mathcal{L}$$

satisfying the Leibniz rules

$$\hat{\nabla}(f\sigma) = \delta f \otimes \sigma + f\hat{\nabla}(\sigma) , \qquad \hat{\overline{\nabla}}(f\sigma) = \overline{\delta} f \otimes \sigma + f\hat{\overline{\nabla}}(\sigma)$$

for all sections σ of $\mathcal{L}$ and f of $\mathcal{B}_X$.

Any (smooth) relative graded connection ∇_r over $\mathcal{L}$ induces a conformal connection, by composition with the projections $\Omega^1_{\mathcal{A}_X/\mathcal{A}_Y} \to \Omega^{1,0}$ or $\Omega^1_{\mathcal{A}_X/\mathcal{A}_Y} \to \Omega^{0,1}$ and with the morphism $\varrho^{p,q} : \Omega^{p,q} \to \mathcal{A}^{p,q}$:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\nabla_r} & \Omega^1_{\mathcal{A}_X/\mathcal{A}_Y} \otimes_{\mathcal{B}_X} \mathcal{L} \\ & \hat\nabla \searrow & \downarrow \\ & & \mathcal{A}^{1,0} \otimes_{\mathcal{B}_X} \mathcal{L} \end{array}, \qquad \begin{array}{ccc} \mathcal{L} & \xrightarrow{\bar\nabla_r} & \Omega^1_{\mathcal{A}_X/\mathcal{A}_Y} \otimes_{\mathcal{B}_X} \mathcal{L} \\ & \hat{\bar\nabla} \searrow & \downarrow \\ & & \mathcal{A}^{0,1} \otimes_{\mathcal{B}_X} \mathcal{L} \end{array}.$$

Given an open cover $\{U_i\}$ of X over which $\mathcal{L}$ trivializes, and nowhere vanishing local sections $\{\sigma_i \in \mathcal{L}(U_i)\}$, we define a set of local *conformal connection (Berezin) forms* $\{\tau_i, \bar\tau'_i\}$ by letting

$$\hat\nabla(\sigma_i) = \tau_i \otimes \sigma_i, \qquad \hat{\bar\nabla}(\sigma_i) = \bar\tau'_i \otimes \sigma_i.$$

Denoting by g_{ij} the transition functions of $\mathcal{L}$, one proves that on $U_i \cap U_j$

$$\tau_j = \tau_i + g_{ij}^{-1}\delta g_{ij}, \qquad \bar\tau'_j = \bar\tau'_i + g_{ij}^{-1}\bar\delta g_{ij};$$

these relations characterize the conformal connection, and permit one to show, by standard partition of unity arguments, the existence of conformal connections.

One may extend $(\hat\nabla, \hat{\bar\nabla})$ to a pair of even morphisms

$$\hat\nabla : \mathcal{A}^{0,1} \otimes_{\mathcal{B}_X} \mathcal{L} \to \mathcal{A}^{1,1} \otimes_{\mathcal{B}_X} \mathcal{L}, \qquad \hat{\bar\nabla} : \mathcal{A}^{1,0} \otimes_{\mathcal{B}_X} \mathcal{L} \to \mathcal{A}^{1,1} \otimes_{\mathcal{B}_X} \mathcal{L},$$

by letting $\hat\nabla(\bar\tau' \otimes \sigma) = \delta(\bar\tau') \otimes \sigma - \bar\tau' \otimes \hat\nabla(\sigma)$ and $\hat{\bar\nabla}(\tau \otimes \sigma) = \bar\delta(\tau) \otimes \sigma - \tau \otimes \hat{\bar\nabla}(\sigma)$.

Definition 4.5. The curvature $\hat K$ of a conformal connection $(\hat\nabla, \hat{\bar\nabla})$ is the morphism

$$\hat K = \hat\nabla \circ \hat{\bar\nabla} + \hat{\bar\nabla} \circ \hat\nabla : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{B}_X} \mathcal{A}^{1,1}.$$

Proposition 4.6. *$\hat K$ is a graded $\mathcal{B}_X$-linear morphism.*

Proof. By direct computation, using lemma 3.11. □

In view of this result, $\hat K$ determines a global section of $\mathcal{A}^{1,1}$, which we denote by $\hat K \in \Gamma(X, \mathcal{A}^{1,1})$ again. If the conformal connection $(\hat\nabla, \hat{\bar\nabla})$ has conformal connection forms $\{\tau_i, \bar\tau'_i\}$, the local expression of $\hat K$ is

$$\hat K_i = \bar\delta \tau_i + \delta \bar\tau'_i.$$

If $K_r \in \Gamma(X, \mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y})$ is the curvature of a relative graded connection ∇_r, an easy computation shows that the curvature $\hat{K} \in \Gamma(X, \mathcal{A}^{1,1})$ of the conformal connection $(\hat{\nabla}, \hat{\bar{\nabla}})$ induced by ∇_r is the image of K_r via the morphism $\Gamma(X, \mathcal{Z}^2_{\mathcal{A}_X/\mathcal{A}_Y}) \to \Gamma(X, \mathcal{A}^{1,1})$ induced by (3.9).

Lemma 4.7. *The image* $[\hat{K}] \in \Gamma(Y, DRB_{X/Y})$ *under the natural morphism* $\Gamma(Y, \pi_* \mathcal{A}^{1,1}) \to \Gamma(Y, DRB_{X/Y})$ *does not depend on the conformal connection.*

Proof. The proof is the usual one: if $(\hat{\nabla}_0, \hat{\bar{\nabla}}_0)$, $(\hat{\nabla}_1, \hat{\bar{\nabla}}_1)$ are two conformal connections, the morphisms

$$\hat{\nabla}_1 - \hat{\nabla}_0 : \mathcal{L} \to \mathcal{A}^{1,0} \otimes_{\mathcal{B}_X} \mathcal{L}, \qquad \hat{\bar{\nabla}}_1 - \hat{\bar{\nabla}}_0 : \mathcal{L} \to \mathcal{A}^{0,1} \otimes_{\mathcal{B}_X} \mathcal{L}$$

are graded $\mathcal{B}_X$-linear and therefore determine global sections $\tau \in \Gamma(X, \mathcal{A}^{1,0}) \simeq \Gamma(Y, \pi_* \mathcal{A}^{1,0})$ and $\bar{\tau}' \in \Gamma(X, \mathcal{A}^{0,1}) \simeq \Gamma(Y, \pi_* \mathcal{A}^{0,1})$; moreover, the morphisms

$$\hat{\nabla}_t = \hat{\nabla}_0 + t\tau, \qquad \hat{\bar{\nabla}}_t = \hat{\bar{\nabla}}_0 + t\bar{\tau}'$$

define a family of conformal connections, whose curvatures are $\hat{K}_t = \hat{K}_0 + t(\bar{\delta}\tau + \delta\bar{\tau}')$. Then, the section $\bar{\delta}\tau + \delta\bar{\tau}' \in \Gamma(X, \mathcal{A}^{1,1})$ lies in the kernel of $\Gamma(X, \mathcal{A}^{1,1}) \to \Gamma(Y, DRB_{X/Y})$, which proves the claim. □

If $[K_r] \in \Gamma(Y, DR^2_{\mathcal{A}_X/\mathcal{A}_Y})$ is the projection of K_r, in view of the commutativity of diagram (3.10) the image of $[K_r]$ in $\Gamma(Y, DRB_{X/Y})$ coincides with $[\hat{K}] \in \Gamma(Y, DRB_{X/Y})$, the image of the curvature $\hat{K}$ of the conformal connection induced by ∇_r. Then, from (4.2) and lemma 4.7 one infers that

$$c_1([\mathcal{L}]) = (\mathrm{i}/2\pi)[\hat{K}] \in \Gamma(Y, DRB_{X/Y}),$$

and by applying the fiberwise Berezin integration one obtains the following result.

Theorem 4.8 (Gauss–Bonnet). *Let* $(X/Y, \mathcal{D})$ *be a SUSY-curve,* $\mathcal{L}$ *a line bundle over* $(X, \mathcal{B}_X)$*, let* $[\mathcal{L}]$ *be its image in* $\mathrm{Pic}(\mathcal{B}_X/\mathcal{B}_Y)$*, and let* $\hat{K}$ *be the curvature of any conformal connection over* $\mathcal{L}$*; then*

$$c_1([\mathcal{L}]) = (\mathrm{i}/2\pi) \int_{\mathcal{A}_X/\mathcal{A}_Y} \hat{K}. \qquad \square$$

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